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Comparative Analysis of Mathematical Models of Penstock Dynamics at Hydropower Plants

Gordana Janevska, Sotir Panovski, Cvete Dimitrieska

Abstract— The paper discusses mathematical modeling of the dynamics of hydraulic circuit (penstock pipe) in hydropower plants. Mathematical models of the hydraulic system (pipeline) in hydropower plants differ in their complexity depending on the assumptions introduced during the model development. Which mathematical model will be applied depends on the specific purpose, as well as on the specifically considered hydropower plant. The paper presents a general nonlinear mathematical model of the penstock pipe, which includes two basic equations that describe the transient phenomena of fluid flow (equation of motion and continuity equation). It also considers the simplified mathematical models in which elastic water column theory, as well as non-elastic water column theory is used. In addition, a simulation scheme for models comparison is presented, and the differences in the dynamics covered by separate mathematical models have been analyzed on the basis of simulation results.

Index Terms— Dynamic systems, Hydropower plants, Mathematical modeling, Penstock.

1 Introduction

IN hydropower plant, the water flow to the turbine inlet through the hydraulic circuit. Characteristics of the hydraulic circuit, i.e. consideration the effects of water inertia, water compressibility, as well as the pipe elasticity, have a major impact on the turbine dynamics. Water inertia causes delay in the changes of flow rate through the turbine with respect to the changes in turbine gate opening. The pipe elasticity entails pressure waves which propagates in the pipe - a phenomenon commonly known as a water hammer. A water hammer occurs by sudden flow rate changes, which causing the pressure changes, above or below the normal pressure. This pressure wave can cause major problems, from noise and vibration to pipe collapse. Under heavy operateing conditions, these effects can lead to damage or destruction of the valves, the turbine guide vanes, as well as the penstock.

2 Mathematical Models for Fluid Flow through a Pipeline

Consider the water flow through the pipeline (fig. 1a). When the gate is suddenly partially closed (flow cross-section area is significantly decreased at the pipeline outlet, that is, the turbine inlet), a pressure (compression) wave is set up and it moves upstream.

Let the pressure p at a slight distance increase by Δp. The equation of motion (Newton's second law) of water in the pipe section is:

$$\left(A\Delta x\rho\right)\frac{dv}{dt} = -A\Delta p \tag{1}$$

where ρ is mass density. The change in pressure in terms of change in head is given by:

$$\Delta p = \rho\ g\ \Delta H$$

where g is acceleration due to gravity. By taking infinitesimally small values of Δx, Δv и Δt, (1) may be written as:

$$\frac{\partial v}{\partial t} = -g\frac{\partial H}{\partial x} \tag{2}$$

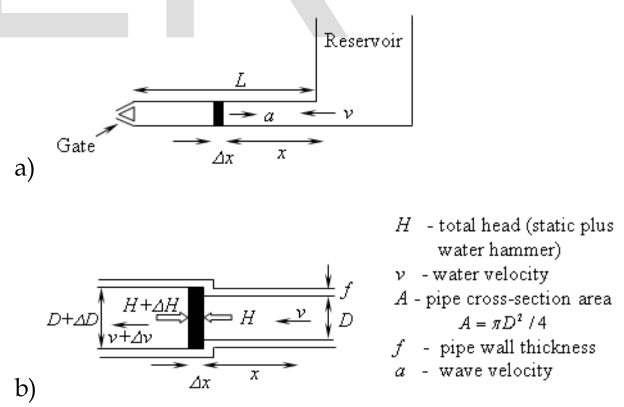

Fig. 1. Water flow through an elastic pipeline: a) Hydraulic system, b) Deformation of the pipe wall at the wave front.

The increase in volume due to pipe wall deformation is:

$$\Delta V_E = \frac{A\ D\ \Delta x}{E\ f}\Delta p \tag{3}$$

where f is thickness of pipe wall, and E is Young's modulus of elasticity of pipe material.

The change in volume of water in the section due to water compressibility is:

$$\Delta V_C = \frac{A\Delta x}{K}\Delta p \tag{4}$$

• Assoc. Prof. Dr.sc Gordana Janevska, Faculty of Technical Sciences, "St.Kliment Ohridski" University - Bitola, Macedonia, E-mail: gordana.janevska@uklo.edu.mk
• Prof. Dr.sc Sotir Panovski, Faculty of Technical Sciences, "St.Kliment Ohridski" University - Bitola, Macedonia, E-mail:panovskisotir@gmail.com
• Prof. Dr.sc Cvete Dimitrieska, Faculty of Technical Sciences, "St.Kliment Ohridski" University - Bitola, Macedonia, E-mail: cvete.stefanovska@uklo.edu.mk

where K is bulk modulus of compression of water.

The increase in mass of water in the considered section of the pipe, which occurs as a consequence of the combined effects of pipe elasticity and water compressibility, is:

$$\Delta m = \rho(\Delta V_E + \Delta V_C) = \rho\, A\, \Delta x \left(\frac{1}{K} + \frac{D}{E\ f}\right)\Delta p \qquad (5)$$

This change should be equal to the change in mass of water within the pipe section during the period Δt determined by the difference between the flow into the section and the flow before this section:

$$\Delta m = \rho\, A\, v\, \Delta t\ - \rho\,(v + \Delta v)\, A\, \Delta t = -\,\rho\, A\, \Delta v\, \Delta t \qquad (6)$$

Equating (5) and (6), and considering infinitesimal small incremental values, the following equation can be obtained:

$$\frac{\partial v}{\partial x} = -\,\alpha\,\frac{\partial H}{\partial t} \qquad (7)$$

where

$$\alpha = \rho\, g \left(\frac{1}{K} + \frac{D}{E\ f}\right) \qquad (8)$$

Equations (2) and (7) are the basic hydraulic equations that determine the flow of a compressible fluid through an elastic pipeline, with friction neglected. They represent a general nonlinear mathematical model of penstock. This mathematical model comprises two basic equations, equation of motion and continuity equation, and describe the transient phenomena of fluid flow through a pipeline. Mathematical model for unsteady flow in penstock is obtained using a one-dimensional approach of modeling.

2.1 Simplifications - Distributed-parameter Model and Lumped-parameter Model

The basic hydraulic equations, that determine the flow of a compressible fluid through an elastic pipe, can be solved by applying the Laplace transform [3]. Expressed in relative units, i.e. (p.u.), in terms of appropriately selected values for the rated head H_{base} and the rated flow Q_{base} as base values, they can be written as:

$$h_2 = h_1 \operatorname{sech}(T_e s) - z_n\, q_2 \tanh(T_e s) \qquad (9)$$

$$q_1 = q_2 \cosh(T_e s) + \frac{1}{z_n} h_2 \sinh(T_e s)\,. \qquad (10)$$

where T_e is the traveling time of the pressure wave (elastic time), in (s):

$$T_e = \frac{\text{penstock length } L}{\text{pressure wave velocity } a}\ , \qquad (11)$$

z_n is the normalized value of the hydraulic surge impedance of the pipeline:

$$z_n = Z_0 \left(\frac{Q_{base}}{H_{base}}\right) = \frac{a}{A\, g}\left(\frac{Q_{base}}{H_{base}}\right) \qquad (12)$$

q is the water flow, and the subscripts 1 and 2 refer to the cross-sections at the beginning and at the end of the pipe, respectively.

The effect of increasing the pressure in a pipe due to rapid flow changes is known as water hammer. Water hammer is a pressure surge or wave caused when a fluid in motion is forced to stop or change direction suddenly. The momentum of the fluid abruptly stopping creates a pressure wave that travels through the media within the pipe system, subjecting everything in that closed system to significant forces. This pressure wave can cause major problems, from noise and vibration to pipe collapse. In hydropower plants this phenomenon is most often caused of:

- rapidly closing the guide vanes (the gate),
- load rejection,
- runaway speed of the turbine due to control system failure.

The speed at which the disturbing pressure waves spread (in an elastic pipe) is determined by the Joukowsky equation:

$$a = \sqrt{\frac{E_F}{\rho}} \cdot \frac{1}{\sqrt{1 + \frac{E_F D}{E_C f}}} = \frac{a_0}{\sqrt{1 + \frac{E_F D}{E_C f}}} \qquad (13)$$

where the wave propagation velocity (m/s) through the water (1425 m/s) in the pipeline is:

$$a_0 = \sqrt{E_F / \rho}$$

and E_F isthe modulus of elasticity of the fluid (for the water E_F =2,03×10^6 kN/m^2).· E_C is the modulus of elasticity of pipe material (for steel pipelines E_C =196,2×10^6 kN/m^2), D is the pipe inner diameter and f is the thickness of pipe wall.

For a rigid (non-elastic) pipeline $E_C \to \infty$, so that $a = a_0$. According to the literature, the speed of propagation of the pressure wave front in the penstock at hydropower plants is usually within the limits: a = 700-1200 m/s.

It should be noted that the wave traveling time (elastic time) T_e is the time required the pressure wave pass one length of the penstock. The time required the positive amplitude of the wave to reach from one to the other pipe end and backwards is $2T_e$, which means that the time required to repeat the entire period (for the positive and negative amplitude of the pressure wave) is $4T_e$.

By analyzing the partial differential equations that define the pressure and flow at each point of the pipeline, a solution for the ratio of pressure (head) and flow in a considered point of the pipeline is obtained. Considering that the penstock outlet (i.e. the entrance into the turbine) is of particular importance for dynamics analysis, under conditions of constant pressure at the upper part of the penstock, this quotient is the ratio of functions in Laplace domain:

$$\frac{h(s)}{q(s)} = -\frac{T_w}{T_e}\frac{1 - e^{-2T_e s}}{1 + e^{-2T_e s}} = -\frac{T_w}{T_e}\tanh(T_e s)\ \cdot \qquad (14)$$

In the above equation, T_w is a time constant of water inertia, often known as the water starting time. This time constant represents the time required the water in the penstock to accelerate from zero velocity to the velocity v_{base} due to the head H_{base} and it is defined by

$$T_w = \frac{L}{Ag}\frac{Q_{base}}{H_{base}} \qquad (15)$$

The transfer function of a distributed-parameter system is obtained by using the approximation:

$$\tanh(T_e s) = sT_e \frac{\prod_{n=1}^{\infty}\left[1+\left(\frac{sT_e}{n\pi}\right)^2\right]}{\prod_{n=1}^{\infty}\left\{1+\left[\frac{2sT_e}{(2n-1)\pi}\right]^2\right\}} \quad . \tag{16}$$

The infinite product expansions of (16) are required to preserve all characteristic roots of the transfer function. Nevertheless, the transfer function may be approximated by lumped-parameter equivalent by retaining an appropriate number of terms of the expansions, depending on the required accuracy and the aim of the study. For example, for n = 4, the transfer function is given by:

$$\frac{h(s)}{q(s)} \approx -sT_w \cdot \frac{1+\left(\frac{T_e}{\pi}\right)^2 s^2}{1+\left(\frac{2T_e}{\pi}\right)^2 s^2} \cdot \frac{1+\left(\frac{T_e}{2\pi}\right)^2 s^2}{1+\left(\frac{2T_e}{3\pi}\right)^2 s^2} \cdot \cdot \frac{1+\left(\frac{T_e}{3\pi}\right)^2 s^2}{1+\left(\frac{2T_e}{5\pi}\right)^2 s^2} \cdot \frac{1+\left(\frac{T_e}{4\pi}\right)^2 s^2}{1+\left(\frac{2T_e}{7\pi}\right)^2 s^2} \quad . \tag{17}$$

2.2 Simplified Mathematical Model - Space Discretization

Another way to solve the system of partial differential equations that describe the dynamics of the fluid flow through a pipeline [4], i.e. the equation of motion and continuity equation, and their reduction on finite dimension is the method of space discretization along the pipe length x. In this case, the partial differential equations describing the flow in the pipeline, for k sections formed by the discretization, will pass into 2k differential equations of the first order. It is often considered justifiable to take the penstock itself as one discrete space element.

In such a case, in fact, the penstock is modeled considering non-compressible fluid, and the rate of flow change in the penstock is determined by equating the rate of change of the water moment with the force caused by water pressure in the penstock, i.e.

$$\rho L \frac{dQ}{dt} = (H_s - H_l - H)A\rho g \tag{18}$$

where Q is the volume flow, L is the length of the pipe, A is the pipe cross section area, g is acceleration due to gravity and ρ is water density. The force of water pressure is determined by the pressure at each end of the penstock: at the penstock inlet it is proportional to the static pressure H_s, while at the guide vanes inlet the force is proportional to the pressure at the turbine entrance H. Due to friction in the pipe, there is also a water friction force represented by the pressure loss H_l. The previous equation can be expressed in per unit form with rated head H_{base} and rated flow Q_{base} as base values. In this case, H_{base} represents the static pressure above the turbine and it is equal to H_s, and Q_{base} represents the flow through the turbine at fully open gate and pressure equal to H_{base}. Dividing both sides of (18) with $H_{base}Q_{base}$, the following equation is obtained:

$$\frac{dq}{dt} = \frac{1}{T_w}(1-h_l-h) \quad , \tag{19}$$

where q=Q/Q_{base} and h=H/H_{base} are in per unit flow and pressure (head), respectively, while T_w=LQ_{base}/AgH_{base} is a time constant of water inertia, often known as the water starting time.

Unsteady flow of real fluid in pipe is always associated with losses of a certain amount of energy, which irreversibly transfer into heat. This process is called dissipation and it is caused by internal friction (viscosity). The lost energy of the fluid flow through a pipe is due to different kinds of resistances that impede the flow.

Friction losses are a complex function of the system geometry, the fluid properties and the flow rate in the system. The head loss due to friction is roughly proportional to the square of the flow rate in most engineering flows (fully developed, turbulent pipe flow). This observation leads to the Darcy-Weisbach equation for head loss due to friction, and for a cylindrical pipe of uniform diameter D and a length L, it is determined by [2]:

$$H_{l1} = \lambda \cdot \frac{L}{D} \cdot \frac{v^2}{2g} \quad , \tag{20}$$

where λ is the friction factor (also called flow coefficient), and v is the mean flow velocity.

The additional losses due to entries and exits, fittings and valves are traditionally referred to as minor losses. These losses represent additional energy dissipation in the flow, usually caused by secondary flows induced by curvature or recirculation. The minor losses are any head loss present in addition to the head loss for the same length of straight pipe. Like friction losses, these losses are roughly proportional to the square of the flow rate [2], i.e.

$$H_{l2} = \xi \cdot \frac{v^2}{2g} \quad , \tag{21}$$

where ξ is loss coefficient. In case of more local resistances, the total head loss is the sum of all of the losses in the length of pipe, each contributing to the overall head loss.

Accordingly, head losses in the pipe are:

$$H_l = \sum_i \xi_i \cdot \frac{v^2}{2g} + \lambda \cdot \frac{L}{D} \cdot \frac{v^2}{2g} = \frac{1}{2g} \cdot \left(\sum_i \xi_i + \lambda \cdot \frac{L}{D}\right) \cdot v^2 = = \frac{1}{2g \cdot A^2} \cdot \left(\sum_i \xi_i + \lambda \cdot \frac{L}{D}\right) \cdot Q^2 \quad .$$

Introducing the denotations:

$$k_1 = \frac{1}{2g} \cdot \left(\sum_i \xi_i + \lambda \cdot \frac{L}{D}\right) \quad \text{and}$$

$$k = \frac{1}{2g \cdot A^2} \cdot \left(\sum_i \xi_i + \lambda \cdot \frac{L}{D}\right) \quad , \tag{22}$$

for a particular case it can be assumed, with sufficient accuracy, that the losses in the penstock are proportional to the square of the velocity, i.e. to the square of the water flow rate. Expressed in per unit form, the head losses equation h_l for the penstock can be written as:

$$h_l = k_f \cdot q^2 \quad . \tag{23}$$

In order to take the sign into account, i.e. the direction of this change in head, (23) is most often expressed through the absolute value of the flow, i.e.:

$$h_l = k_f \cdot |q| \cdot q \quad . \tag{24}$$

After the space discretization of the continuity equation, taking into account only one discrete space element, it is obtained:

$$\dot{h} = -\kappa\,(q_2 - q_1) \quad , \tag{25}$$

where

$$\kappa = \left(\frac{\pi}{2}\right)^2 \frac{a^2}{gAL}\frac{Q_{base}}{H_{base}} \quad , \tag{26}$$

$h = h_{\mathrm{PC}}$ dynamic pressure at the penstock outlet, in per unit,
$q_1 = q_{\mathrm{PC}}$ water flow at the penstock inlet, in per unit,
$q_2 = q_{\mathrm{T}}$ water flow at the penstock outlet, in per unit.

The described penstock model presented by (19) and (25) is nonlinear due to the quadratic term of the pressure losses h_l. If the flow q through the penstock is denoted by q_{PC}, after the linearization, the mathematical model of the penstock dynamics in a state space is:

$$\begin{bmatrix} \Delta\dot{q}_{\mathrm{PC}} \\ \Delta\dot{h}_{\mathrm{PC}} \end{bmatrix} = \begin{bmatrix} -\dfrac{2k_{\mathrm{PC}}}{T_w} q_{\mathrm{PC0}} & -\dfrac{1}{T_w} \\ \kappa & 0 \end{bmatrix} \begin{bmatrix} \Delta q_{\mathrm{PC}} \\ \Delta h_{\mathrm{PC}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\kappa \end{bmatrix} \Delta q_{\mathrm{T}} \quad . \tag{27}$$

The relationship between the pressure and the flow is of the greatest interest, and the transfer function in s -space is:

$$\frac{\Delta h_{\mathrm{PC}}(s)}{\Delta q_{\mathrm{T}}(s)} = -\kappa \frac{\dfrac{2k_{\mathrm{PC}}}{T_w} q_{\mathrm{PC0}} + s}{\dfrac{\kappa}{T_w} + \dfrac{2k_{\mathrm{PC}}}{T_w} q_{\mathrm{PC0}} s + s^2} \quad , \tag{28}$$

where Δh_{PC} is the dynamic pressure in penstock (deviation from the operateing point) in per unit; Δq_{PC} is the water flow at the penstock inlet (deviation from the operateing point) in per unit; Δq_{T} is the water flow in (p.u.) at the penstock outlet (deviation from the operating point); q_{PC0} is the water flow in (p.u.) for the operating point.

According to the model (28), it can be seen that the linearized dynamics in the penstock at a certain operating point depends only on the flow q_{PC0}. Analyzing the poles of (28), conclusions about the penstock dynamics can be drawn.

The transfer function (28) also can be obtained by using second-order Padé approximation of the term $e^{-2T_e s}$ in (14)

$$\frac{1 - sT_e + \dfrac{T_e^2}{3}s^2}{1 + sT_e + \dfrac{T_e^2}{3}s^2} \quad . \tag{29}$$

If the losses in the pipeline are also neglected ($k_{\mathrm{PC}} = 0$), then the transfer function (28) is equal to the one obtained with the previous procedure

$$\frac{\Delta h_{\mathrm{PC}}(s)}{\Delta q_{\mathrm{T}}(s)} = -\frac{sT_w}{1 + \left(\dfrac{2}{\pi}T_e\right)^2 s^2} \quad . \tag{30}$$

In case of a short penstock, sometimes it is justifiable to use even more simpler model of the penstock dynamics. In that case, it is assumed that the water is non-compressible and the pipe is inelastic, i.e. the first-order Padé approximation of the term $e^{-2T_e s}$ in (14) is used:

$$\frac{1 - sT_e}{1 + sT_e} \quad , \tag{31}$$

and it is obtained:

$$\frac{\Delta h_{\mathrm{PC}}(s)}{\Delta q_{\mathrm{T}}(s)} = -sT_w \quad , \tag{32}$$

which is an expression that is often used in older literature on turbine regulation.

3 Comparison of the Different Mathematical Models of Penstock Dynamics

Which one of the presented mathematical models of the penstock dynamics will be used, it depends on the purpose, as well as on the specially considered hydroelectric power plant. For more detailed dynamics analysis, the model with distributed parameters (with space discretization) can be used. In order to analyze the dynamics of turbine governor, low-order models can be used, taking into account the impact of losses in the pipeline. For the control algorithm synthesis, it is sufficient to use a model of even lower order (sometimes even the first order model), without taking into consideration the impact of losses in the pipeline. Later, in the phase of verification of the designed control algorithm, a more-compex model is usually used in order to cover as many real physical influences as possible.

A simulation scheme for comparing the dynamics of the different penstock models is given in fig. 2.

The parameters of one specific hydropower plant are used to performed the simulations. Pressure responses for different mathematical models for the same input (flow rate) are given in fig. 3. Linear models are considered in the small region around the operating point.

The symbols used on the block diagram in fig. 2 and on the graphs given in fig. 3 have the following meaning:

hn0 - first order approximation for pressure rise in penstock (rigid pipe, incompressible water);

hn1, hn2, hn3 and hn4 - approximations according to (17) respectively for n=1, n=2, n=3 and n=4;

q - input (excitation) in the penstock system, actually it is water flow at the penstock outlet;

qPC - water flow at the penstock inlet;

hPC - approximation of pressure increase in the penstock according to (19) and (25).

By comparing the responses on the same input (change in flow rate) for the different models, the differences in dynamics of the separate mathematical models can be seen. Only deviations from the operating point were considered. The first order mathematical model (32), then the second order model (28) and the model with distributed parameters (17) for n = 1, 2, 3 and 4, have been compared.

From the responses shown in fig. 3, where the second order model given by (19) and (25) are compared with the model given by (17), it can be concluded that larger differences in the response of pressure rise occur in the case of (32), which presents a derivative term (hn0). Regarding the other cases of pressure increase, there are oscillations that completely overlap in the first harmonic, and there are differences in higher

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order models due to the influence of higher order harmonics.

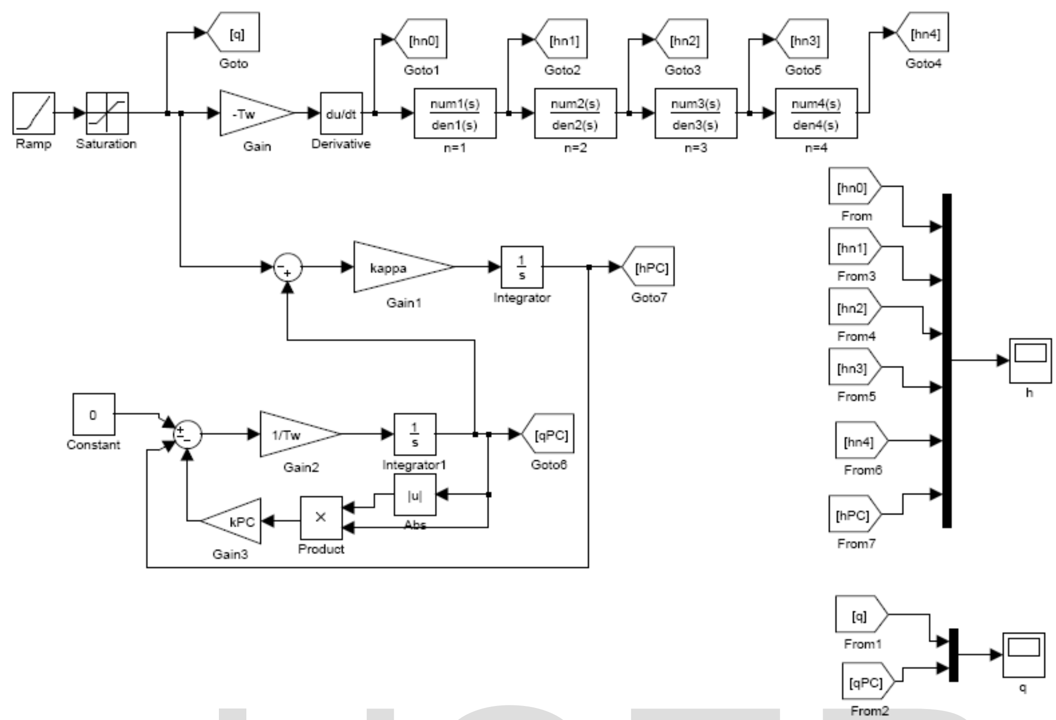

Fig. 2. Simulation scheme for comparing the dynamics of the different penstock models.

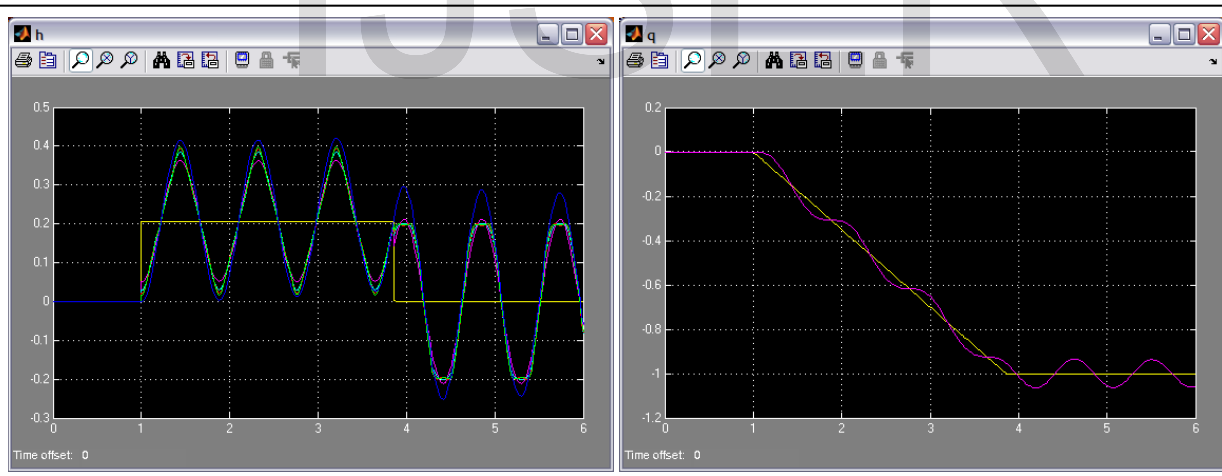

Fig. 3. Pressure responses for different models for the same input (flow rate). Linear models are considered in the small region around the operating point. The symbols on the left graph are: (yellow) hn0, (pink) hn1, (turquoise) hn2, (red) hn3, (green) hn4, (blue) hPC. The symbols on the right graph are: (yellow) q, (pink) qPC.

4 Conclusion

In hydro power plant, the water flow into the turbine through the hydraulic circuit. Characteristics of the hydraulic circuit, i.e. consideration the effects of water inertia, water compressibility, as well as the pipe elasticity, have a major impact on the turbine dynamics.

The paper discusses mathematical modeling of the dynamics of hydraulic circuit (penstock pipe) in hydropower plants. Mathematical models of the hydraulic circuit in hydropower plants differ in their complexity depending on the assumptions introduced during the model development. The paper presents a general nonlinear mathematical model of the penstock pipe. It also considers the simplified mathematical models in which elastic water column theory, as well as non-elastic

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water column theory is used.

In addition, a simulation scheme for model comparison is presented. The simulation is performed using the parameters of a real hydropover plant. The differences in the penstock dynamics covered by separate mathematical models have been analyzed on the basis of the simulation results.

Which one of the presented mathematical models of the penstock dynamics will be used, it depends on the purpose, as well as on the specially considered hydroelectric power plant.

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